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A Brunn–Minkowski inequality for the Monge–Ampère eigenvalue

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Abstract

We prove a Brunn–Minkowski-type inequality for the eigenvalue Λ of the Monge–Ampère operator: $\Lambda^{-1/2n}$ is concave in the class of C^2_+ domains in $\mathbb{R}^n$ endowed with Minkowski addition. The equality case is explicitly described too. The main device of the proof is a notion of addition for convex functions, called *infimal convolution*, which corresponds to the Minkowski addition of the graphs of the involved functions.

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1. Introduction

The present paper is devoted to proving the Brunn–Minkowski inequality for the eigenvalue

$$\Lambda(\Omega) = \inf \left\{ \frac{-\int_\Omega u \det(D^2 u)\, \mathrm{d}x}{\int_\Omega |u|^{n+1}\, \mathrm{d}x} : \begin{array}{l} u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) \text{ is non-zero} \\ \text{and convex on } \Omega,\ u = 0 \text{ on } \partial\Omega \end{array} \right\} \tag{1}$$

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of the Monge–Ampère operator, where Ω is a C^2_+ domain in $\mathbb{R}^n$ (i.e. a bounded open set with C^2 boundary $\partial\Omega$ which has everywhere positive Gauss curvature) and D^2u denotes the Hessian matrix of u.

Equivalently, $\Lambda(\Omega)$ can be defined as the only positive constant Λ such that the following eigenvalue problem has a (strictly) convex solution:

$$\begin{cases} \det(D^2u) = (-1)^n \Lambda\, u^n & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \quad u < 0 & \text{in } \Omega. \end{cases} \tag{2}$$

Notice that $\Lambda(\cdot)$ is positively homogeneous of degree $-2n$ with respect to dilatation of sets, i.e.

$$\Lambda(t\Omega) = t^{-2n}\Lambda(\Omega) \quad \text{for } t > 0. \tag{3}$$

Indeed, if u solves (2) for Ω, then, by setting $u_t(x) = u(x/t)$, we have

$$\det(D^2u_t(x)) = t^{-2n}\det(D^2u(x/t)) = (-1)^n t^{-2n}\Lambda(\Omega)\, u_t(x)^n \quad \text{in } t\Omega$$

and (3) follows.

For details about the eigenvalue of Monge–Ampère, we refer mainly to [18,21]; in particular, in Section 3 of [18] some applications and an interesting stochastic interpretation of Λ are presented.

The original form of the Brunn–Minkowski inequality involves volumes of convex bodies (i.e. compact convex sets with non-empty interior) and states that $V(\cdot)^{1/n}$ is a concave function with respect to Minkowski addition, where $\mathrm{V}(\cdot)$ denotes n-dimensional Lebesgue measure and Minkowski addition of convex sets is defined as follows.

Definition 1.1. Let $\lambda \in [0,1]$ and let Ω_0 and Ω_1 be convex subsets of $\mathbb{R}^n$; we define their *Minkowski linear combination* by

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1 = \left\{(1-\lambda)\,x_0 + \lambda\,x_1 \;:\; x_i \in \Omega_i \;,\; i = 0,1\right\}. \tag{4}$$

With this notation, the Brunn–Minkowski inequality for volumes reads

$$\mathrm{V}(K_\lambda)^{\frac{1}{n}} \geqslant (1-\lambda)\,\mathrm{V}(K_0)^{\frac{1}{n}} + \lambda\,\mathrm{V}(K_1)^{\frac{1}{n}} \tag{5}$$

for convex bodies K_0 and K_1 and $\lambda \in [0,1]$.

Inequality (5) is one of the fundamental results in the theory of convex bodies and several other important inequalities, e.g. the isoperimetric inequality, can be deduced from it. It can be extended to measurable sets and it holds also, with the right exponent, for the other quermassintegrals. We refer the interested reader to [20] and to the survey paper [14] for this topic.

It is interesting to notice that analogues of (5) hold for many set functionals: for instance, there are Brunn–Minkowski inequalities for electrostatic capacity (see [1,6]) and for p-capacity (see [11]), for the transfinite diameter (see [2,9], for an extension), for the first eigenvalue of the Laplacian (see [4,5]), for the Poincaré constant (see [10]) and for torsional rigidity (see [3]); furthermore, extensions of the case of torsional rigidity are in [8,10].

The main purpose of this paper is to prove the analogue of (5) for Λ, as stated in the following theorem.

Theorem 1. *Let Ω_0 and Ω_1 be n-dimensional C^2_+ domains and* $\lambda \in [0,1]$. *Then*

$$\Lambda(\Omega_\lambda)^{-1/2n} \geqslant (1-\lambda)\,\Lambda(\Omega_0)^{-1/2n} + \lambda\,\Lambda(\Omega_1)^{-1/2n}. \tag{6}$$

In other words, (6) says that $\Lambda(\cdot)^{-\frac{1}{2n}}$ is a concave function on the class of C^2_+ domains endowed with the Minkowski addition (4). Notice that the exponents, $-\frac{1}{2}n$ in (6) and $1/n$ in (5), are determined by the degree of homogeneity of the set functions involved.

The main ingredient of the proof of Theorem 1 is a notion of convex combination of functions called *infimal convolution*. If Ω_0 and Ω_1 are two convex bounded open sets in $\mathbb{R}^n$ and $u_i \in C(\overline{\Omega}_i)$ is convex in Ω_i and vanishes on $\partial\Omega_i$, $i = 0, 1$, for $x \in \overline{\Omega}_\lambda$ we define

$$\begin{aligned} u_\lambda(x) = \min\{&(1-\lambda)u_0(x_0) + \lambda\, u_1(x_1) : \\ &(1-\lambda)\, x_0 + \lambda\, x_1 = x\,,\ \ x_i \in \overline{\Omega}_i,\ i = 0, 1\}. \end{aligned} \tag{7}$$

Roughly speaking, u_λ is the convex function whose epigraph is the Minkowski linear combination of the epigraphs of u_0 and u_1.

We will not prove Theorem 1 directly, but instead prove the following result.

Theorem 2. *With the assumptions of Theorem 1 and with the same notation, we have*

$$\Lambda(\Omega_\lambda) \leqslant (1-\lambda)\Lambda(\Omega_0) + \lambda\Lambda(\Omega_1), \tag{8}$$

i.e. $\Lambda(\cdot)$ is convex with respect to Minkowski addition.

A standard calculation shows that, thanks to the homogeneity of $\Lambda(\cdot)$, Theorems 2 and 1 are equivalent.

In Brunn–Minkowski-type inequalities, the discussion of the equality case often has its own relevance, since it can be used to prove uniqueness in related Minkowski problems (see for instance [6,16,17]). Hence, we state separately the case of equality.

Theorem 3. *Equality holds in* (6) *if and only if* Ω_1 *is homothetic to* Ω_0.

Notice that the proof of the latter is, in fact, a direct consequence of the equality case in the Prékopa–Leindler inequality.

2. Preliminaries

2.1. Basic notation

If $a, b \in \mathbb{R}^n$, we denote by $\langle a, b\rangle$ their scalar product and we denote by $|a|$ the usual norm of the vector a, i.e. $|a| = \sqrt{\langle a, a\rangle}$.

By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$, that is $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. If $x_0 \in \mathbb{R}^n$ and $r > 0$, $B(x_0, r)$ is the open ball centered at x_0 with radius r, that is $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

For points or vectors $p \in \mathbb{R}^{n+1}$, we will often emphasize the dependence on the last coordinate, by setting $p = (x, t)$ with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

For $A, B \subseteq \mathbb{R}^n$, we say that B is *homothetic* to A if there exist $\xi \in \mathbb{R}^n$ and $\alpha > 0$ such that $B = \alpha(\xi + A) = \{\alpha(\xi + x) : x \in A\}$.

Throughout the paper, Ω and K, possibly with subscripts, will be convex subsets of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. If not otherwise specified, Ω will be a C^2_+ set, which means a bounded open set with C^2 boundary $\partial\Omega$ which has everywhere positive Gauss curvature, while K will be reserved to denote a *convex body*, that is a compact convex set, with non-empty interior.

Let $p \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$; we set

$$H_{p,\alpha} = \{x \in \mathbb{R}^n : \langle x, p\rangle = \alpha\} \quad \text{and} \quad H^-_{p,\alpha} = \{x \in \mathbb{R}^n : \langle x, p\rangle \leqslant \alpha\}.$$

Let K be a convex body in $\mathbb{R}^n$. We say that p is an *exterior normal vector* of K at x_0 if $x_0 \in K \cap H_{p,\alpha}$ and $K \subseteq H^-_{p,\alpha}$; in such a case, we say also that the hyperplane $H_{p,\alpha}$ is a *support hyperplane* and that $H^-_{p,\alpha}$ is a *supporting halfspace* (with exterior normal vector p) of K.

If M is an $n \times n$ symmetric matrix, we denote by $\mathrm{tr}(M)$ and $\det(M)$ its trace and its determinant, respectively. We recall that the function $\det(M)^{1/n}$ is concave in the class of symmetric non-negative definite matrices.

If u is twice differentiable, by ∇u and D^2u we denote, as usual, the gradient of u and its Hessian matrix, respectively, i.e. $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ and $D^2u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$.

If a, b are real positive numbers, $\alpha \in [-\infty, +\infty]$ and $\lambda \in (0, 1)$, we define

$$m_\alpha(a, b, \lambda) = \begin{cases} [(1-\lambda)a^\alpha + \lambda b^\alpha]^{1/\alpha} & \text{if } \alpha \in (-\infty, 0) \cup (0, +\infty), \\ \min(a, b) & \text{if } \alpha = -\infty, \\ a^{1-\lambda} b^\lambda & \text{if } \alpha = 0, \\ \max(a, b) & \text{if } \alpha = +\infty. \end{cases}$$

We recall that *Jensen's inequality for means* implies that

$$m_\alpha(a, b, \lambda) \leqslant m_\beta(a, b, \lambda) \quad \text{if } \alpha \leqslant \beta. \tag{9}$$

In particular, the *arithmetic–geometric mean inequality* holds

$$a^{1-\lambda} b^\lambda \leqslant (1-\lambda)a + \lambda b \quad \text{for every } \ a, b \geqslant 0, \quad \lambda \in [0, 1].$$

2.2. About Brunn–Minkowski-type inequalities

Here, we prove that (6) and (8) are equivalent. Indeed, (6) is equivalent to

$$\Lambda(\Omega_\lambda) \leqslant [(1-\lambda)\, \Lambda(\Omega_0)^{-1/2n} + \lambda\, \Lambda(\Omega_1)^{-1/2n}]^{-2n},$$

which implies (8) thanks to (9) with $\alpha = -\frac{1}{2}n$ and $\beta = 1$ (or, equivalently, by the convexity of t^{-2n} for $t > 0$). Furthermore, notice that this shows that equality in (8) forces equality in (6).

Conversely, assume that (8) holds (for every $\lambda \in (0, 1)$ and for every couple of C^2_+ sets) and let

$$\mu = \frac{\Lambda_1^{-1/2n}}{\Lambda_0^{-1/2n} + \Lambda_1^{-1/2n}} \quad \left(\text{hence } \ 1 - \mu = \frac{\Lambda_0^{-1/2n}}{\Lambda_0^{-1/2n} + \Lambda_1^{-1/2n}}\right),$$

where $\Lambda_i = \Lambda(\Omega_i)$, $i = 0, 1$. Let $C_i = \Lambda_i^{1/2n}\Omega_i$, $i = 0, 1$; then

$$(1-\mu)C_0 + \mu C_1 = \frac{1}{\Lambda_0^{-1/2n} + \Lambda_1^{-1/2n}}(\Omega_0 + \Omega_1)$$

and (8), applied to C_0, C_1 and μ, reads

$$\Lambda((1-\mu)C_0 + \mu C_1) \leqslant 1,$$

i.e.

$$\Lambda(\Omega_0 + \Omega_1)^{-1/2n} \geqslant \Lambda(\Omega_0)^{-1/2n} + \Lambda(\Omega_1)^{-1/2n}.$$

To obtain (6), just replace Ω_0 with $(1-\lambda)\Omega_0$ and Ω_1 with $\lambda\Omega_1$.

2.3. About the Prékopa–Leindler inequality

The Prékopa–Leindler inequality is an integral version of (5); we refer to [14] for a good presentation of both formulas and their connections. The classical form of this inequality is stated in the following theorem.

Theorem 4 (*Prékopa–Leindler inequality*). *Let $\lambda \in (0,1)$ and let f, g and h be non-negative integrable functions on $\mathbb{R}^n$. Assume that*

$$h(z) \geqslant f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y, z \in \mathbb{R}^n$ such that $z = (1-\lambda)x + \lambda y$; then

$$\int_{\mathbb{R}^n} h(z)\,\mathrm{d}z \geqslant \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)\,\mathrm{d}y \right)^{\lambda}. \tag{10}$$

The equality case is settled in Theorem 12 of [12] and it can be stated in the following way.

Lemma 5. *If equality holds in (10), then there exist $m > 0$ and $b \in \mathbb{R}^n$ such that*

$$f(x) = \frac{\int f\,dx}{\int g\,dx}\, m^n g(mx + b)$$

for almost every $x \in \mathbb{R}^n$.

2.4. About differentiability of convex functions

We recall here some well-known facts about differentiability of convex functions, that will be often used in the following. For more details, we refer to [19] or to [20, Section 1.5].

Let Ω be a convex open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a convex function. Then u is continuous on Ω and it is Lipschitz continuous on every compact subset of Ω; hence it is differentiable almost everywhere on Ω.

A relevant feature of convex functions is that the notion of differential has a natural extension which is defined in every point $x \in \Omega$, even in that ones where u is not differentiable: the set

$$\partial u(x) = \{ v \,:\, u(y) \geqslant u(x) + \langle v, y - x \rangle \ \forall y \in \Omega \}, \tag{11}$$

is called the *subdifferential* of u at x; a vector $v \in \partial u(x)$ is called a *subgradient* of u at x. The subdifferential $\partial u(x)$ is a non-empty closed convex set for every $x \in \Omega$.

Let $x \in \Omega$; a vector $v \in \mathbb{R}^n$ belongs to $\partial u(x)$ if and only if the vector $(v, -1) \in \mathbb{R}^{n+1}$ is an exterior normal vector to $\operatorname{epi} u$ at $(x, u(x))$, where $\operatorname{epi} u$ is the *epigraph* of u, i.e.

$$\operatorname{epi} u = \{(x, t) \in \mathbb{R}^{n+1} : x \in \Omega,\ t \geqslant u(x)\}.$$

Furthermore, u is differentiable at x if and only if it has a unique subgradient at x; in such a case $\partial u(x) = \{\nabla u(x)\}$. Finally, we recall that if u is (convex and) differentiable in Ω, then u is, in fact, of class C^1 in Ω (see [19, Theorem 25.5]).

2.5. About support functions of convex bodies

The *support function* of a convex body K is defined by

$$h_K(p) = \sup_{x \in K} \langle p, x \rangle, \qquad p \in \mathbb{R}^n. \tag{12}$$

If Ω is a bounded convex open subset of $\mathbb{R}^n$, by h_Ω we mean $h_{\overline{\Omega}}$. Notice that the supremum in (12) is attained, by the compactness of K, and this happens at a boundary point of K; hence we can write $h_K(p) = \max_{x \in \partial K} \langle p, x \rangle$ for every $p \in \mathbb{R}^n$.

It is often convenient to consider h_K as a function from S^{n-1} to $\mathbb{R}$; on the other hand, if $p \in \mathbb{R}^n$ and $\theta = p/|p| \in S^{n-1}$, it holds trivially $h_K(p) = |p| h_K(\theta)$.

The support function has an easy geometric meaning: if $p \in \mathbb{R}^n \setminus \{0\}$, then $h_K(p) = \alpha$ if and only if $H^-_{p,\alpha}$ is the supporting halfspace of K with exterior normal vector p.

In the following lines, we collect a list of well-known properties of support functions of convex bodies, for which we refer mainly to Section 1.7 of [20].

First of all, we notice that, for every convex body K, h_K is a sublinear function. Actually, there is a one-to-one correspondence between sublinear functions and convex bodies, since for every sublinear function $h : \mathbb{R}^n \to \mathbb{R}$ there is a unique convex body K with support function h.

Support functions are obviously non-decreasing with respect to inclusion, i.e. $h_K(\cdot) \leqslant h_L(\cdot)$ if and only if $K \subseteq L$. Minkowski linear combination of convex bodies is equivalent to linear combination of the corresponding support functions, i.e.

$$h_{(1-\lambda)K+\lambda L}(\cdot) = (1-\lambda) h_K(\cdot) + \lambda h_L(\cdot). \tag{13}$$

If we set

$$H(K, p) = H_{p, h_K(p)}, \quad H^-(K, p) = H^-_{p, h_K(p)} \quad \text{and} \quad F(K, p) = H(K, p) \cap K,$$

for two convex bodies K and L and $p \in \mathbb{R}^n \setminus \{0\}$ we have [20, Theorem 1.7.5])

$$\begin{aligned} H(K + L, \cdot) &= H(K, \cdot) + H(L, \cdot), \\ F(K + L, \cdot) &= F(K, \cdot) + F(L, \cdot). \end{aligned} \tag{14}$$

For our convenience, we assemble Theorem 1.7.4 and Corollary 1.7.3 of [20] in the following proposition.

Proposition 6. *Let K be a convex body in $\mathbb{R}^n$ and $p \in \mathbb{R}^n \setminus \{0\}$, then the subdifferential of h_K at p is precisely the support set $F(K, p)$. Hence, h_K is differentiable at p if and only if $F(K, p)$ contains only one point x and in such a case*

$$\nabla h_K(p) = x. \tag{15}$$

Notice that the above proposition implies that $h_K \in C^1(\mathbb{R}^n \setminus \{0\})$ if and only if K is strictly convex. Furthermore, if K is of class C^2_+, let $v_K : \partial K \to S^{n-1}$ be the *Gauss map* of K, i.e. $v_K(x)$ is the outward unit normal vector of K at x; then v_K is of class C^1 with inverse map $v_K^{-1} \in C^1(S^{n-1})$, $h_K \in C^2(\mathbb{R}^n \setminus \{0\})$ and $\nabla h_K(p) = v_K^{-1}(p/|p|)$ for every $p \in \mathbb{R}^n \setminus \{0\}$ (see [20, Section 2.5, for more details]).

2.6. About the Monge–Ampère operator

The Dirichlet problem for the Monge–Ampère equation has been treated by many authors; let us just recall [7] and refer the reader to [15] for further references.

Here, we are mainly concerned with the eigenvalue problem (2), which has been firstly treated by Lions [18]. There the author proves that (2) has a solution pair (u, Λ), $u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ strictly convex, $\Lambda > 0$, which is unique up to a scalar multiplication of u.

The variational characterization (1) of the eigenvalue Λ was investigated by Tso [21] and then by Wang [22], who dealt with the eigenvalue problem and the related variational theory for a class of elliptic equations (the so-called *Hessian equations*), including the Monge–Ampère equation.

Let us recall here the geometric interpretation of the Monge–Ampère equation and the related concept of *generalized solution* (mainly due to Alexandrov and Bakelmann). Essentially, it relies on the fact that, if u is a C^2 convex function in Ω, for every Borel set $\beta \subseteq \Omega$

$$\int_\beta \det(D^2u)\,\mathrm{d}x$$

is the measure of the set

$$\chi_u(\beta) = \{\nabla u(x) : x \in \beta\},$$

the image of β through the gradient map $\nabla u(\cdot)$. On the other hand, if u is convex, even at a point where it is not differentiable, it is possible to consider its subdifferential $\partial u(x)$. Hence, to every Borel set $\beta \subseteq \Omega$, it is possible to associate the set

$$\chi_u(\beta) = \cup_{x\in\beta}\, \partial u(x), \tag{16}$$

which results to be a Borel set too; we denote by $\omega(u, \beta)$ its n-dimensional Lebesgue measure. The measure $\omega(u, \cdot)$ on Ω so defined is called *the Monge–Ampère measure* associated to u. A Monge–Ampère equation,

$$\det(D^2 u) = \psi(x) \quad \text{in } \Omega, \tag{17}$$

has thus a natural measure interpretation as

$$\omega(u, \cdot) = \mu(\cdot), \tag{18}$$

where $\mu(\beta) = \int_\beta \psi(x)\,\mathrm{d}x$ for every Borel set $\beta \subseteq \Omega$. A convex function which solves (18) is called a *generalized solution* (or *Alexandrov solution*) of (17) (and it is a classical solution when it is regular enough). Of course (18) can be considered for every non-negative (even not absolutely continuous) measure μ. We recall also that if u_k, $k \in \mathbb{N}$, are convex functions in Ω such that $u_k \to u$ uniformly on compact subsets of Ω, then the associated Monge–Ampère measures $\omega(u_k, \cdot)$ converge weakly to $\omega(u, \cdot)$; in particular, it can be useful to recall here Proposition 1.1 of [21].

Proposition 7. *Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of convex functions in $C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$. Assume that*: (i) *$u_k = 0$ on $\partial\Omega$ for every k*; (ii) *∇u_k is uniformly bounded*; (iii) *u_k converge uniformly to u in Ω. Then $-\int_\Omega u\,\mathrm{d}\omega(u, x)$ is finite and $-\int_\Omega u_k\,\mathrm{d}\omega(u_k, x)$ converges to $-\int_\Omega u\,\mathrm{d}\omega(u, x)$ as $k \to \infty$.*

For more details about generalized solutions to Monge–Ampère equation and about the Monge–Ampère measure associated to a convex function, we refer the reader to [15, Chapter 1], for example.

Let u be a C^1 strictly convex function in Ω, then the mapping $x \to \nabla u(x)$ is a continuous one-to-one mapping between Ω and $\chi_u(\Omega)$, with continuous inverse $(\nabla u)^{-1}$ (which, by the way, coincides with the gradient map ∇u^* of the conjugate u^* of u, see Section 6). In fact, even if u is not of class C^1, the strict convexity implies that the mapping

$$\nabla u^* : \chi_u(\Omega) \to \Omega,$$

which assigns to every $\xi \in \chi_u(\Omega)$ the unique $x \in \Omega$ such that $\xi \in \partial u(x)$, is well defined and continuous and we can say that the measure $\omega(u, \cdot)$ over Ω is the so called *push-forward* by ∇u^* of the Lebesgue measure $\mathcal{L}^n$ over $\chi_u(\Omega)$, (and we write $\omega(u, \cdot) = \nabla u^*_\# \mathcal{L}^n(\cdot)$). Hence, by Theorem 2.4.18 of [13], we have

$$\int_\Omega f(x)\,\mathrm{d}\omega(u, x) = \int_{\chi_u(\Omega)} f(\nabla u^*(\xi))\,\mathrm{d}\xi \tag{19}$$

for any $f \in C(\Omega)$.

3. The infimal convolution of convex functions

Let $\lambda \in [0,1]$, let Ω_0 and Ω_1 be two *strictly convex bounded open sets* in $\mathbb{R}^n$ and let

$$\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1.$$

Furthermore, from now on, we will assume that $u_i \in C(\overline{\Omega}_i)$ *is a strictly convex function which vanishes on* $\partial\Omega_i$, $i = 0, 1$.

The intersections of the epigraphs of u_0 and u_1 with the halfspace $\{(x,t) \in \mathbb{R}^{n+1} : t \leqslant 0\}$ define two convex bodies in $\mathbb{R}^{n+1}$:

$$K_i = \{(x,t) \in \mathbb{R}^{n+1} : x \in \overline{\Omega}_i,\ u_i(x) \leqslant t \leqslant 0\} \quad i = 0, 1.$$

Notice that ∂K_i, $i = 0, 1$, can be divided in two parts: the graph of u_i over $\overline{\Omega}_i$ and the set Ω_i (intended as the n-dimensional subset of $\mathbb{R}^{n+1}$ defined by $\{(x,0) \in \mathbb{R}^n \times \mathbb{R} : x \in \Omega_i\}$), that is

$$\partial K_i = \Omega_i \cup \{(x, u_i(x)) : x \in \overline{\Omega}_i\} \quad i = 0, 1.$$

Let us consider the Minkowski sum K_λ of K_0 and K_1 in $\mathbb{R}^{n+1}$: it is a convex body contained in the halfspace $\{(x,t) \in \mathbb{R}^{n+1} : t \leqslant 0\}$, such that $K_\lambda \cap \{t = 0\} = \overline{\Omega}_\lambda$.

Definition 3.1. We define u_λ as the function whose epigraph (intersected with $\{t \leqslant 0\}$) is K_λ, i.e.

$$K_\lambda = \{(x,t) \in \mathbb{R}^{n+1} : x \in \overline{\Omega}_\lambda,\ u_\lambda(x) \leqslant t \leqslant 0\}.$$

Equivalently, we could have said that the graph of u_λ over $\overline{\Omega}_\lambda$ is given by $\partial K_\lambda \setminus \Omega_\lambda$.

It is immediate to verify that u_λ coincides with the infimal convolution of u_0 and u_1 as defined in (7). Indeed

$$\begin{aligned} u_\lambda(x) &= \min\{t : (x,t) \in K_\lambda\} \\ &= \min\{(1-\lambda)t_0 + \lambda t_1 : (x_i, t_i) \in K_i,\ i = 0, 1,\ x = (1-\lambda)x_0 + \lambda x_1\} \\ &= \min\{(1-\lambda)t_0 + \lambda t_1 : x_i \in \overline{\Omega}_i,\ u_i(x_i) \leqslant t_i \leqslant 0, x = (1-\lambda)x_0 + \lambda x_1\} \\ &= \min\{(1-\lambda)u_0(x_0) + \lambda u_1(x_1) : x_i \in \overline{\Omega}_i, x = (1-\lambda)x_0 + \lambda x_1\}. \end{aligned}$$

It is also easily seen that $u_\lambda(x) \in C(\overline{\Omega}_\lambda)$ and it vanishes if and only if $x \in \partial\Omega_\lambda$, that is

$$u_\lambda = 0 \ \text{ on } \partial\Omega_\lambda \quad \text{and} \quad u_\lambda < 0 \ \text{ in } \Omega_\lambda.$$

Furthermore, the following lemma is almost straightforward.

Lemma 8. *For every $x \in \overline{\Omega}_\lambda$ there exists a unique couple of points $(x_0, x_1) \in \overline{\Omega}_0 \times \overline{\Omega}_1$ such that*

$$x = (1-\lambda)x_0 + \lambda x_1 \quad \textit{and} \quad u_\lambda(x) = (1-\lambda)u_0(x_0) + \lambda u_1(x_1). \tag{20}$$

Moreover, the function u_λ is strictly convex in $\overline{\Omega}_\lambda$.

Proof. The first assertion is easily inferred by (7) and the strict convexity of u_0 and u_1. Indeed, the function $(x_0, x_1) \to (1-\lambda)u_0(x_0) + \lambda u_1(x_1)$ is strictly convex in the convex subset of $\overline{\Omega}_0 \times \overline{\Omega}_1$ consisting of points (x_0, x_1) such that $x = (1-\lambda)x_0 + \lambda x_1$; thus its minimum is unique.

Now, let $x, y, z \in \overline{\Omega}_\lambda$, $y \neq z$, $\mu \in (0,1)$, such that $x = (1-\mu)y + \mu z$. There exist $(x_0, x_1), (y_0, y_1), (z_0, z_1) \in \overline{\Omega}_0 \times \overline{\Omega}_1$, such that

$$\begin{aligned} x &= (1-\lambda)x_0 + \lambda x_1, \quad u_\lambda(x) = (1-\lambda)u_0(x_0) + \lambda u_1(x_1), \\ y &= (1-\lambda)y_0 + \lambda y_1, \quad u_\lambda(y) = (1-\lambda)u_0(y_0) + \lambda u_1(y_1), \\ z &= (1-\lambda)z_0 + \lambda z_1, \quad u_\lambda(z) = (1-\lambda)u_0(z_0) + \lambda u_1(z_1). \end{aligned}$$

By letting $\xi_i = (1-\mu)y_i + \mu z_i$, $i = 0, 1$, we have $x = (1-\lambda)\xi_0 + \lambda \xi_1$. Notice that, since $y \neq z$, either $y_0 \neq z_0$ or $y_1 \neq z_1$. Hence

$$\begin{aligned} u_\lambda(x) &\leqslant (1-\lambda)u_0(\xi_0) + \lambda u_1(\xi_1) \\ &= (1-\lambda)u_0((1-\mu)y_0 + \mu z_0) + \lambda u_1((1-\mu)y_1 + \mu z_1) \\ &< (1-\lambda)[(1-\mu)u_0(y_0) + \mu u_0(z_0)] + \lambda[(1-\mu)u_1(y_1) + \mu u_1(z_1)] \\ &= (1-\mu)u_\lambda(y) + \mu u_\lambda(z), \end{aligned}$$

which proves the strict convexity of u_λ. $\square$

In order to investigate the differentiability of u_λ and the relationship between the gradient map of u_λ and the gradients of u_0 and u_1, we have to further exploit the geometric meaning of infimal convolution.

Proposition 9. *The subgradient image of Ω_λ through u_λ is the union of the subgradient images of Ω_0 and Ω_1 through u_0 and u_1, respectively; i.e. the following holds*

$$\chi_{u_\lambda}(\Omega_\lambda) = \chi_{u_0}(\Omega_0) \cup \chi_{u_1}(\Omega_1). \tag{21}$$

Proof. Let $\chi_i = \chi_{u_i}(\Omega_i)$ and $F_i(\cdot) = F(K_i, \cdot)$, $i = 0, \lambda, 1$. Then, by (14), we have

$$F_\lambda(\theta) = (1-\lambda)F_0(\theta) + \lambda F_1(\theta) \quad \text{for every } \theta \in S^n.$$

When $\theta = (0, \ldots, 0, 1)$, this simply reduces to $\overline{\Omega}_\lambda = (1-\lambda)\overline{\Omega}_0 + \lambda \overline{\Omega}_1$.

If $S^n \ni \theta \neq (0, \dots, 0, 1)$, by the strict convexity assumption on u_i and Ω_i (and the consequent strict convexity of u_λ and Ω_λ), we have that $F_i(\theta)$ is made by a single point $p_i \in \partial K_i \setminus \Omega_i$, that is $p_i = (x_i, u(x_i))$ for some $x_i \in \overline{\Omega}_i$, $i = 0, \lambda, 1$. Then

$$F_\lambda(\theta) = \{(x_\lambda, u_\lambda(x_\lambda))\} = \{((1-\lambda)x_0 + \lambda x_1, (1-\lambda)u_0(x_0) + \lambda u_1(x_1))\},$$

which implies that (x_0, x_1) is the couple of points associated to x_λ by (20).

Let $\xi \in \chi_0 \cup \chi_1$ and let $\theta \in S^n$ be defined by

$$\theta = \frac{1}{\sqrt{1 + |\xi|^2}} (\xi, -1).$$

Then, for $i = 0, \lambda, 1$, $F_i(\theta) = \{(x_i, u(x_i))\}$ for some $x_i \in \overline{\Omega}_i$, which implies that $(\xi, -1)$ is an exterior normal vector to K_i at $(x_i, u_i(x_i))$ and this, if $x_i \in \Omega_i$, is equivalent to $\xi \in \partial u_i(x_i)$. Furthermore, we have that (x_0, x_1) is the couple of points associated to x_λ by (20), as we have seen above. Notice that, since either $\xi \in \chi_0$ or $\xi \in \chi_1$, then either $x_0 \in \Omega_0$ or $x_1 \in \Omega_1$, which implies that $x_\lambda \in \Omega_\lambda$. Then, $\xi \in \partial u_\lambda(x_\lambda) \subset \chi_\lambda$. So, we have proved that $\chi_0 \cup \chi_1 \subseteq \chi_\lambda$.

Conversely, if $\xi \in \chi_\lambda$, then $\xi \in \partial u_\lambda(x_\lambda)$ for some $x_\lambda \in \Omega_\lambda$. Let $(x_0, x_1) \in \overline{\Omega}_0 \times \overline{\Omega}_1$ be the couple of points associated to x_λ by (20). Hence we have $F_i(\theta) = \{(x_i, u_i(x_i))\}$, $i = 0, 1$, by the uniqueness of (x_0, x_1). Since $x_\lambda \in \Omega_\lambda$, we have $u_\lambda(x_\lambda) < 0$ and then, by (20), either $x_0 \in \Omega_0$ or $x_1 \in \Omega_1$, which yields that either $\xi \in \partial u_0(x_0)$ or $\xi \in \partial u_1(x_1)$. Hence, the reverse inclusion is proved too. $\square$

Lemma 10. *If u_i is of class $C^1(\Omega_i)$, for $i = 0, 1$, then u_λ is of class $C^1(\Omega_\lambda)$.*

Proof. By the properties of convex functions, we have only to prove that u_λ is differentiable in Ω_λ, which is equivalent to say that $\partial u_\lambda(x)$ contains one vector only (namely $\nabla u_\lambda(x)$) for every $x \in \Omega_\lambda$.

Assume by contradiction that $\xi, \eta \in \partial u_\lambda(x)$ for some $x \in \Omega_\lambda$ and some $\xi \neq \eta$. Notice that, by strict convexity, we have $F_\lambda(\xi, -1) = F_\lambda(\eta, -1) = \{(x, u_\lambda(x))\}$. Let $(x_0, x_1) \in \overline{\Omega}_0 \times \overline{\Omega}_1$ be the unique couple of points associated to x by (20) and let $y_i, z_i \in \overline{\Omega}_i$ such that $F_i(\xi, -1) = \{(y_i, u_i(y_i))\}$ and $F_i(\eta, -1) = \{(z_i, u_i(z_i))\}$, $i = 0, 1$. Then, by (14) and by the uniqueness of the couple (x_0, x_1), we have $x_0 = y_0 = z_0$ and $x_1 = y_1 = z_1$. Since $x_\lambda \in \Omega_\lambda$, at least one of the points x_0 and x_1 is internal to the corresponding Ω_i; say, by instance, $x_0 \in \Omega_0$. Then it should be $\xi = \nabla u_0(x_0) = \eta$, which is impossible.

Remark 11. A direct consequence of Proposition 9 is that, if $u_0 \in C^{0,1}(\overline{\Omega}_0)$ and $u_1 \in C^{0,1}(\overline{\Omega}_1)$, then $u_\lambda \in C^{0,1}(\overline{\Omega}_\lambda)$; indeed, Lipschitz continuity is equivalent, for a convex function u, to the boundedness of the subgradient image $\chi_u(\Omega)$.

Remark 12. Notice that, the arguments of the proof of Proposition 9, allow us to say something more on the relationship between a point $x \in \Omega_\lambda$ and the points x_0 and x_1

associated to it by Lemma 8. Indeed, let $x \in \Omega_\lambda$ and let $\xi \in \partial u_\lambda(x)$; then there are exactly three possibilities:

(i) $\xi \in \chi_0 \cap \chi_1$: then $x_i = \nabla u_i^*(\xi) \in \Omega_i$ for $i = 0, 1$; if u_0 and u_1 are regular enough, this is equivalent to $\nabla u_\lambda(x) = \nabla u_0(x_0) = \nabla u_1(x_1)$.

(ii) $\xi \in \chi_0 \setminus \chi_1$: then $x_0 = \nabla u_0^*(\xi) \in \Omega_0$, while $x_1 = \nabla h_{\Omega_1}(\xi) \in \partial\Omega_1$ (furthermore, we have $\nabla u_\lambda(x) = \nabla u_0(x_0)$, when the involved functions are regular enough).

(iii) $\xi \in \chi_1 \setminus \chi_0$: then $x_1 = \nabla u_1^*(\xi) \in \Omega_1$, while $x_0 = \nabla h_{\Omega_0}(\xi) \in \partial\Omega_0$ (furthermore, we have $\nabla u_\lambda(x) = \nabla u_1(x_1)$, when the involved functions are regular enough).

4. Proof of Theorem 1

As we said in the Introduction, we will prove Theorem 1 in the equivalent form stated in Theorem 2.

For $i = 0, 1$, let Ω_i be a C^2_+ domain and $u_i \in C^\infty(\Omega_i) \cap C^{1,1}(\overline{\Omega}_i)$ be the (strictly) convex solution of

$$\begin{cases} \det(D^2 u_i) = (-1)^n \Lambda(\Omega_i)\, u_i^n & \text{in } \Omega_i, \\ u_i = 0 \quad \text{on } \partial\Omega_i, \quad u_i < 0 & \text{in } \Omega_i, \end{cases} \tag{22}$$

normalized in such a way that

$$\int_{\Omega_i} |u_i|^{n+1} = 1.$$

Let u_λ be defined by Definition 3.1 (or by (7)); then, as we have seen in the previous section, u_λ is strictly convex in Ω_λ and of class $C^1(\Omega_\lambda) \cap C^{0,1}(\overline{\Omega}_\lambda)$. Hence, it is easily seen that u_λ can be uniformly approximated by a sequence of convex functions in $C^2(\Omega_\lambda) \cap C^{0,1}(\overline{\Omega}_\lambda)$ which vanish on $\partial\Omega_\lambda$ and with uniformly bounded gradients; thanks to Proposition 7 and to the definition (1) of $\Lambda(\Omega_\lambda)$, this yields

$$\Lambda(\Omega_\lambda) \leqslant \frac{-\int_{\Omega_\lambda} u_\lambda(x)\omega(u_\lambda, \mathrm{d}x)}{\int_{\Omega_\lambda} |u_\lambda|^{n+1}\, \mathrm{d}x}. \tag{23}$$

Here, $\omega(u_\lambda, \cdot)$ is the Monge–Ampère measure associated to u_λ, as defined in Section 2.3. Theorem 2 is then a straightforward consequence of the following estimate:

$$\frac{-\int_{\Omega_\lambda} u_\lambda(x)\omega(u_\lambda, dx)}{\int_{\Omega_\lambda} |u_\lambda|^{n+1}\, \mathrm{d}x}$$
$$\leqslant (1-\lambda)\left(\frac{-\int_{\Omega_0} u_0 \det(D^2 u_0)\, \mathrm{d}x}{\int_{\Omega_0} |u_0|^{n+1}\, \mathrm{d}x}\right) + \lambda\left(\frac{-\int_{\Omega_1} u_1 \det(D^2 u_1)\, \mathrm{d}x}{\int_{\Omega_1} |u_1|^{n+1}\, \mathrm{d}x}\right),$$

which is, in turn, a straightforward consequence of the following Lemma 13 and Proposition 14.

Lemma 13. *Let u_0 and u_1 be two convex functions defined on the open bounded convex sets Ω_0 and Ω_1, respectively. Assume that, for $i = 0, 1$, $u_i \in C(\overline{\Omega_i})$ and $u_i = 0$ on $\partial\Omega_i$; moreover, assume that $\int_{\Omega_0} |u_0|^{n+1} = \int_{\Omega_1} |u_1|^{n+1} = 1$. Let u_λ be the infimal convolution of u_0 and u_1, for some $\lambda \in (0, 1)$. Then*

$$\int_{\Omega_\lambda} |u_\lambda|^{n+1}\,\mathrm{d}x \geqslant 1. \tag{24}$$

Proof. Notice that Brunn–Minkowski inequality (5) immediately gives

$$\int_{\Omega_\lambda} |u_\lambda| \geqslant m_{\frac{1}{n+1}}\left(\int_{\Omega_0} |u_0|\,\mathrm{d}x, \int_{\Omega_1} |u_1|\,\mathrm{d}x, \lambda\right),$$

which coincides with (3.5) of [5] for $\alpha = 1$ (and with Theorem 11 of [12] for $p(u, v) = u + v$). But here we are interested in $\int |u|^{n+1}\,\mathrm{d}x$. By (7), we have

$$\begin{aligned}|u_\lambda(x)| &= -u_\lambda(x)\\ &= \max\{(1-\lambda)|u_0(x_0)| + \lambda\,|u_1(x_1)| \,:\, x_i \in \overline{\Omega}_i,\ x = (1-\lambda)x_0 + \lambda x_1\}.\end{aligned}$$

By the arithmetic–geometric mean inequality, this implies

$$|u_\lambda(x)| \geqslant \max\{|u_0(x_0)|^{1-\lambda}|u_1(x_1)|^\lambda \,:\, x_i \in \overline{\Omega}_i,\ x = (1-\lambda)x_0 + \lambda x_1\}.$$

By setting $v_i(x) = |u_i(x)|^{n+1}$ if $x \in \Omega_i$ and $v_i(x) = 0$ if $x \in \mathbb{R}^n \setminus \Omega_i$, for $i = 0, \lambda, 1$, we clearly have

$$v_\lambda(x) \geqslant v_0(x_0)^{1-\lambda} v_1(x_1)^\lambda \quad \text{whenever } x = (1-\lambda)x_0 + \lambda\,x_1,$$

then (24) is a direct consequence of Theorem 4. □

Proposition 14. *Let $u_i \in C^1(\Omega_i) \cap C^{0,1}(\overline{\Omega}_i)$ be a strictly convex function which vanishes on $\partial\Omega_i$, $i = 0, 1$. Then*

$$-\int_{\Omega_\lambda} u_\lambda\,\mathrm{d}\omega(u_\lambda, x) = (1-\lambda)\left(-\int_{\Omega_0} u_0\,\mathrm{d}\omega(u_0, x)\right) + \lambda\left(-\int_{\Omega_1} u_1\,\mathrm{d}\omega(u_1, x)\right).$$

Proof. By (19) and Lemma 8, the statement is equivalent to

$$\int_{\chi_\lambda} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi = (1-\lambda)\int_{\chi_0} u_0((\nabla u_0)^{-1}(\xi))\,\mathrm{d}\xi + \lambda\int_{\chi_1} u_1((\nabla u_1)^{-1}(\xi))\,\mathrm{d}\xi,$$

where we shortened $\chi_{u_i}(\Omega_i)$ by χ_i, for $i = 0, \lambda, 1$.

By Lemma 9, $\chi_\lambda = \chi_0 \cup \chi_1$; hence we can write

$$\begin{aligned}\int_{\chi_\lambda} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi = &\int_{\chi_0\cap\chi_1} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi\\ &+\int_{\chi_0\setminus\chi_1} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi\\ &+\int_{\chi_1\setminus\chi_0} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi.\end{aligned}$$

Consider the first one of the three integrals in the right-hand side of the above formula. For $\xi \in \chi_0 \cap \chi_1$, let

$$x = (\nabla u_\lambda)^{-1}(\xi), \quad x_0 = (\nabla u_0)^{-1}(\xi), \quad x_1 = (\nabla u_1)^{-1}(\xi).$$

Then $x \in \Omega_\lambda$, $x_i \in \Omega_i$ for $i = 0, 1$ and, as we have seen in the previous section, (x_0, x_1) is exactly the couple of points associated to x by (20), which can be rewritten as

$$\begin{aligned}(\nabla u_\lambda)^{-1}(\xi) &= (1-\lambda)(\nabla u_0)^{-1}(\xi) + \lambda(\nabla u_1)^{-1}(\xi),\\ u_\lambda((\nabla u_\lambda)^{-1}(\xi)) &= (1-\lambda)u_0((\nabla u_0)^{-1}(\xi)) + \lambda u_1((\nabla u_1)^{-1}(\xi)).\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\chi_0\cap\chi_1} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi\\ &\quad= \int_{\chi_0\cap\chi_1} [(1-\lambda)u_0((\nabla u_0)^{-1}(\xi)) + \lambda u_1((\nabla u_1)^{-1}(\xi))]\,\mathrm{d}\xi\\ &\quad= (1-\lambda)\int_{\chi_0\cap\chi_1} u_0((\nabla u_0)^{-1}(\xi))\,\mathrm{d}\xi + \lambda\int_{\chi_0\cap\chi_1} u_1((\nabla u_1)^{-1}(\xi))\,\mathrm{d}\xi \qquad (25)\end{aligned}$$

Now, let $\xi \in \chi_0 \setminus \chi_1$ and let

$$x = (\nabla u_\lambda)^{-1}(\xi), \quad x_0 = (\nabla u_0)^{-1}(\xi), \quad x_1 = \nabla h_{\Omega_1}(\xi).$$

Then $x \in \Omega_\lambda$, $x_0 \in \Omega_0$, $x_1 \in \partial\Omega_1$ and we have again that x, x_0 and x_1 satisfy (20). In this case $u_1(x_1) = 0$, hence we have

$$u_\lambda((\nabla u_\lambda)^{-1}(\xi)) = (1-\lambda)u_0((\nabla u_0)^{-1}(\xi)),$$

which yields

$$\int\limits_{\chi_0\setminus\chi_1} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi = (1-\lambda)\int\limits_{\chi_0\setminus\chi_1} u_0((\nabla u_0)^{-1}(\xi))\,\mathrm{d}\xi. \tag{26}$$

Analogously we infer

$$\int\limits_{\chi_1\setminus\chi_0} u_\lambda((\nabla u_\lambda)^{-1}(\xi))\,\mathrm{d}\xi = \lambda\int\limits_{\chi_1\setminus\chi_0} u_1((\nabla u_1)^{-1}(\xi))\,\mathrm{d}\xi. \tag{27}$$

Putting toghether (25)–(27), the proof is completed. $\square$

Notice that C^1 regularity of the involved functions is completely unnecessary in the proof of the above proposition; the statement still holds simply assuming u_0 and u_1 strictly convex in the respective domains and the proof remains true word by word by simply replacing $(\nabla u_i)^{-1}$ with ∇u_i^* for $i = 0, \lambda, 1$.

5. The equality case

In this section, we prove Theorem 3.

If Ω_1 is homothetic to Ω_0, then equality holds in (6) by the homogeneity of Λ and by its invariance with respect to translation.

Conversely, if equality holds in (6), then the arguments of Section 2.2. show that equality must hold in (8), up to a normalization of the involved sets; precisely, this means that we should have

$$\Lambda((1-\mu)C_0 + \mu C_1) = 1 = (1-\mu)\Lambda(C_0) + \mu\Lambda(C_1),$$

where $\mu = \frac{\lambda\Lambda_1^{-1/2n}}{(1-\lambda)\Lambda_0^{-1/2n}+\lambda\Lambda_1^{-1/2n}}$, $C_i = \Lambda_i^{1/2n}\Omega_i$ and $\Lambda_i = \Lambda(\Omega_i) > 0$, $i = 0, 1$ (notice that $\mu \in (0, 1)$ if $\lambda \in (0, 1)$).

Theorem 3 is then a corollary of the following.

Theorem 15. *Equality holds in* (8) *if and only if* Ω_1 *is a translate of* Ω_0.

Proof. Clearly, if Ω_0 is a translate of Ω_1, equality holds in (8).

Conversely, by Proposition 14, Lemma 13 and (23), if equality holds in (8), then

$$\int_{\Omega_\lambda} |u_\lambda|^{n+1}\,\mathrm{d}x = 1.$$

By Lemma 5, the latter implies

$$u_0(x) = m^{n/(n+1)} u_1(mx + b) \tag{28}$$

for some $m > 0$ and some $b \in \mathbb{R}^n$. Here u_i is the solution of (2) for Ω_i, $i = 0, 1$, normalized in such a way that $\int_{\Omega_i} |u_i|^{n+1}\,\mathrm{d}x = 1$, and u_λ is the infimal convolution of u_0 and u_1. (28) immediately gives $\Omega_1 = m\Omega_0 + b$. We finally notice that, in fact, the dilatation's factor m must be 1; indeed, let $\Omega_1 = m\Omega_0$, then $(1-\lambda)\Omega_0 + \lambda\Omega_1 = [(1-\lambda) + \lambda m]\Omega_0$ and the equality case of (8) reads

$$[(1-\lambda) + \lambda m]^{-2n} \Lambda(\Omega_0) = (1-\lambda)\Lambda(\Omega_0) + \lambda m^{-2n} \Lambda(\Omega_0),$$

which implies

$$[(1-\lambda) + \lambda m]^{-2n} = (1-\lambda) + \lambda m^{-2n}.$$

By the strict convexity of the function $f(t) = t^{-2n}$ (for $t > 0$), the latter, for $\lambda \in (0, 1)$, is true if and only if $m = 1$. $\square$

6. Some remarks

Let $\Omega \subset \mathbb{R}^n$ be a convex bounded open set and let $u \in C(\overline{\Omega})$ be a convex function. The *conjugate* of u is defined as follows

$$u^*(\xi) = \max\{\langle \xi, x\rangle - u(x) \,:\, x \in \overline{\Omega}\} \qquad \text{for } \xi \in \mathbb{R}^n. \tag{29}$$

As it is a supremum of linear functions, u^* is obviously a convex function.

For details about conjugates of convex functions, we refer again to [19] (mostly to Sections 12 and 26 therein); here we just point out some properties connected with our result.

Notice that, if we set $K_u = \operatorname{epi} u$ (or also $K_u = \operatorname{epi} u \cap \{(x, t) \in \mathbb{R}^{n+1} \,:\, t \leqslant 0\}$), then we have

$$u^*(\xi) = h_{K_u}(\xi, -1) \qquad \text{for every } \xi \in \mathbb{R}^n, \tag{30}$$

where h_{K_u} is the support function of K_u; furthermore we notice that $h_{K_u}(\xi, -1) = h_\Omega(\xi)$ if $\xi \in \mathbb{R}^n \setminus \chi_u(\Omega)$, hence u^* is sublinear out of $\chi_u(\Omega)$.

If Ω is strictly convex and u is strictly convex in Ω, then (30) and Proposition 6 imply that u^* is differentiable (hence C^1) on $\mathbb{R}^n$. Furthermore, $\nabla u^*(\xi)$ is equal to the point $x \in \overline{\Omega}$ such that $(\xi, -1)$ is an exterior normal vector to K_u at $(x, u(x))$ and we have $F(K_u, (\xi, -1)) = \{(x, u(x))\}$; if $\xi \in \chi_u(\Omega)$ this means that $\nabla u^*(\xi)$ is the point $x \in \Omega$ such that $\xi \in \partial u(x)$. Hence, the notation introduced in Section 2.6 for the reverse subdifferential mapping is consistent with the notation here used for the conjugate function. If u is strictly convex and differentiable in Ω, we can finally write

$$\nabla u^* = (\nabla u)^{-1} \quad \text{in } \chi_u(\Omega). \tag{31}$$

We will consider $\chi_u(\Omega)$ as the natural domain for u^* and, from now on, when we speak of u^*, *we always mean its restriction to* $\chi_u(\Omega)$.

Eq. (31) implies that $u \in C^2_+(\Omega)$ (i.e. u is of class C^2 and D^2u is positive definite in Ω) if and only if $u^* \in C^2_+(\chi_u(\Omega))$; moreover, in such a case, we have

$$D^2u^*(\xi) = D^2u(x)^{-1} \quad \text{where } x = \nabla u^*(\xi) \quad \text{and} \quad \xi = \nabla u(x). \tag{32}$$

The infimal convolution of convex functions behaves particularly well under conjugation, see [19, Section 26]. Indeed, let u_0, u_1 and u_λ be as in the previous section: by Definition 3.1, (30) and (13) we have

$$u^*_\lambda(\xi) = (1-\lambda)u^*_0(\xi) + \lambda u^*_1(\xi) \quad \text{for every } \xi \in \chi_0 \cap \chi_1. \tag{33}$$

By the concavity of $\det^{1/n}(\cdot)$ in the class of positive semidefinite symmetric matrices and by the arithmetic–geometric mean inequality, we get

$$\begin{aligned}
\det D^2u^*_\lambda &= \det((1-\lambda)D^2u^*_0 + \lambda D^2u^*_1) \\
&\geqslant [(1-\lambda)(\det D^2u^*_0)^{1/n} + \lambda(\det D^2u^*_1)^{1/n}]^n \\
&\geqslant (\det D^2u^*_0)^{1-\lambda}(\det D^2u^*_1)^\lambda \quad \text{in } \chi_0 \cap \chi_1.
\end{aligned}$$

The latter, together with (32), proves at one time that $u_\lambda \in C^2(\nabla u^*_\lambda(\chi_0 \cap \chi_1))$ and that

$$\det D^2u_\lambda(x) \leqslant \det D^2u_0(x_0)^{1-\lambda} \det D^2u_1(x_1)^\lambda \tag{34}$$

for every $x \in \nabla u^*_\lambda(\chi_0 \cap \chi_1)$, where $(x_0, x_1) \in \nabla u^*_0(\chi_0 \cap \chi_1) \times \nabla u^*_1(\chi_0 \cap \chi_1)$ is the couple of points associated to x by (20). Since u_i is the solution of (22), for $i = 0, 1$, (34) can be rewritten as

$$\det D^2u_\lambda(x) \leqslant (\Lambda_0|u_0(x_0)|^n)^{1-\lambda}(\Lambda_1|u_1(x_1)|^n)^\lambda \tag{35}$$

which yields, again by arithmetic–geometric mean inequality and by (20),

$$\det D^2 u_\lambda \leqslant \Lambda_0^{1-\lambda} \Lambda_1^{\lambda} |u_\lambda|^n \qquad \text{in } \nabla u_\lambda^*(\chi_0 \cap \chi_1). \tag{36}$$

Notice that, if we were able to obtain the pointwise estimate (36) in the whole of Ω_λ, we would easily obtain (6); indeed, by multiplying by $-u_\lambda$, then integrating over Ω_λ both sides and taking in account (1), we would obtain $\Lambda(\Omega_\lambda) \leqslant \Lambda_0^{1-\lambda}\Lambda_1^{\lambda}$, which is equivalent to (6) by standard homogeneity arguments. Pointwise estimates like (36) are exploited in [11,8,10] to obtain Brunn–Minkowski inequalities for other functionals. On the other hand, in this case, it is not possible to prove (36) out of $\nabla u_\lambda^*(\chi_0 \cap \chi_1)$, if χ_0 and χ_1 do not coincide; the same (35) proves that $\det D^2 u_\lambda(x) \to 0$ as $x \to \partial(\nabla u_\lambda^*(\chi_0 \cap \chi_1))$ and this make us unable to obtain the $C^2(\Omega_\lambda)$ regularity for u_λ. The best we could say is that u_λ is of class C^2 in $\Omega_\lambda \setminus \nabla u^*(\gamma)$, where $\gamma = \partial(\chi_0 \cap \chi_1) \cap \chi_\lambda$.

Finally, a remark about the regularity of the involved sets: throughout the paper, Ω_0 and Ω_1 (and, consequently, Ω_λ) have been assumed to be C^2_+ domains, but it is quite natural to ask whether this requirement can be weakened and if the result here presented is true for general convex sets. The reason for the C^2_+ assumption is that we refer to [18,21] for the definition of Λ and in both the papers the involved set is required to be so regular. As far as the author knows, there does not exist in literature a definition of $\Lambda(\Omega)$ and a related theory which avoid this assumption and this task is beyond the aim of this paper.

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References

[1] C. Borell, Capacitary inequalities of the Brunn–Minkowski type, Math. Ann. 263 (1984) 179–184.
[2] C. Borell, Hitting probability of killed Brownian motion: a study on geometric regularity, Ann. Sci. Ecole Norm. Sup. Paris 17 (1984) 451–467.
[3] C. Borell, Greenian potentials and concavity, Math. Ann. 272 (1985) 155–160.
[4] C. Borell, Diffusion equations and geometric inequalities, Potential Anal. 12 (2000) 49–71.
[5] H.J. Brascamp, E.H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, and with an application to the diffusion equation, J. Funct. Anal. 22 (1976) 366–389.
[6] L.A. Caffarelli, D. Jerison, E.H. Lieb, On the case of equality in the Brunn–Minkowski inequality for capacity, Adv. Math. 117 (1996) 193–207.
[7] L.A. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. I Monge–Ampère equation, Comm. Pure Appl. Math. 37 (1984) 369–402.
[8] A. Colesanti, Brunn–Minkowski inequalities for variational functionals and related problems, Adv. Math., to appear.
[9] A. Colesanti, P. Cuoghi, The Brunn–Minkowski inequality for the *n*-dimensional logarithmic capacity, Potential Anal., to appear.

[10] A. Colesanti, P. Cuoghi, P. Salani, Brunn–Minkowski inequalities for two functionals involving the *p*-Laplace operator, Appl. Anal., to appear (special issue dedicated to Carlo Pucci).
[11] A. Colesanti, P. Salani, The Brunn–Minkowski inequality for *p*-capacity of convex bodies, Math. Ann. 327 (2003) 459–479.
[12] S. Dubuc, Critères de convexité et inégalités intégrales, Ann. Inst. Fourier 27 (1) (1977) 135–165.
[13] H. Federer, Geometric Measure Theory, Grundlehren der Mathematischen Wissenschaften, vol. 153, Springer, New York, 1969.
[14] R.J. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002) 355–405.
[15] C.E. Gutiérrez, The Monge–Ampère Equation, Progress in Nonlinear Differential equations and their Applications, vol. 44, Birkhäuser, Boston, 2001.
[16] D. Jerison, A Minkowski problem for electrostatic capacity, Acta Math. 176 (1996) 1–47.
[17] D. Jerison, The direct method in the calculus of variations for convex bodies, Adv. Math. 122 (1996) 262–279.
[18] P.L. Lions, Two remarks on Monge–Ampere equations, Ann. Mat. Pura Appl. 142 (4) (1985) 262–275.
[19] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton NJ, 1970.
[20] R. Schneider, Convex Bodies: the Brunn–Minkowski Theory, Cambridge University Press, Cambridge, 1993.
[21] K. Tso, On a real Monge–Ampère functional, Invent. Math. 101 (1990) 425–448.
[22] X.-J. Wang, A class of fully nonlinear elliptic equations and related functionals, Indiana Univ. Math. J. (1) 43 (1994) 25–54.